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Estimates for Functional Partial Differential Equations

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Abstract

In this presentation we consider a class of reaction-diffusion equations under initial and boundary conditions and with nonlinear reaction terms containing a functional of type "maxima". By assuming that the initial density as well the boundary data are Hölder continuous, and reaction functions have different rates we give two stability criteria. We extend the existence and uniqueness result for the parabolic equation with delay to the case with "maxima". The uniqueness and asymptotic behavior of the solutions are discussed as well. The above mentioned equations are used for mathematical simulation in theoretical physics, thermodynamics, chemistry, mechanics of materials, biology, ecology, etc.

Keywords: *Reaction-diffusion equation; parabolic equation; stability; "maxima"; asymptotic behavior.*

1 Introduction.

There are lots of mathematical models of evolutionary processes using parabolic partial differential equations (PDE) or called reaction-diffusion equations of the form

$$u_t - Lu = F \quad in \quad D, \tag{1}$$

where u is unknown function, L is an uniformly elliptic operator, $D = (0, T) \times \Omega$, and $\Omega \subset \mathbb{R}^n$ is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, [4], [6]. The reaction function $F = F(t, x, u)$ depends continuously on the arguments t - time, $x = (x_1, x_2, \dots, x_n)$ - space, and the unknown function $u = u(t, x)$. Such parabolic equations can be subjected to certain boundary and initial conditions (see e.g. [4]). The boundary condition $Bu = h(t, x)$ is defined by the boundary operator $B \equiv \alpha_0(t, x)\frac{\partial}{\partial \nu} + \beta_0(t, x)$, where $\alpha_0(t, x)$, $\beta_0(t, x)$ and $h(t, x)$ are nonnegative Hölder continuous functions on the boundary ∂D. The initial condition is given by an initial function $u_0(x) \equiv u(t_0, x)$ in Ω defined at the initial time t_0, which can be taken $t_0 =$

0. Thus we pose the initial and boundary value problem (IBVP). There has been increasing interest in the reaction-diffusion equations of type (1) during past few decades. We do emphasize that plenty of papers and monographs devoted to these problems have been published so far. The qualitative theory to these equations as existence, uniqueness, oscillation, stability and blow-up is already developed in details. We encounter mathematical models of evolution processes arising in different areas which contain PDEs with delay (deviating), i.e. their reaction functions F has the form $F(t, x, u(t, x), u(t - \sigma, x))$, [1, 2, 9]. Here the unknown function u depends smoothly on the time moment $t - \sigma$, where the delay (deviation) of the time is expressed by $\sigma > 0$. In other words the unknown function u is taken in a position at σ units back, that is to say as though the equation under consideration has past memory. Of special interest is the problem connected with the existence of blow-up solutions, [7].

In our presentation there is a more general mathematical model of type (1) with a reaction function of the form $F(t, x, u(t, x), \max_{s\in[t-\sigma,t]} u(s, x)) \equiv f(t, x, u(t, x)) + R(t, x, \max_{s\in[t-\sigma,t]} u(s, x))$, [2, 3, 6]. These equations are known as parabolic PDEs with "maxima". Here F depends not only on u taken in the instantaneous time t and space x, but also on the function $\max_{s\in[t-\sigma,t]} u(s, x)$ defined in the time interval $[t - \sigma, t]$ that begins at $t - \sigma$ and continues to t, and $t \in [-\sigma, T)$ for some positive number T that in some cases could be replaced by infinity. Then the domain of existence of the PDE must be taken as $D_{-\sigma} = [-\sigma, T) \times \Omega$. The results of this paper provide explicit analytical information about existence, uniqueness and stability of the solutions for parabolic PDEs with "maxima".

In lots of applications of these equations the "maxima" is applied when the control corresponds to the maximal deviation of the regulated quantity that could be for instance temperature, heat, current density, pressure and so on. Meanwhile, the study of differential equations with "maxima" continue in several directions - existence and uniqueness of the solutions, oscillation, stability, asymptotic behavior of the solutions etc. The oscillation properties of the solutions of the ordinary differential equations with "maxima" were studied by Bainov and his group of associates (see e.g., [1, 2, 5], and the references given there). The theory of neutral partial differential equations of hyperbolic and parabolic type with "maxima" was represented for the first time in the monograph of Bainov and Mishev [1]. More interesting results of existence, uniqueness, oscillation, asymptotic behavior of the solutions of PDEs with "maxima" can be seen in the same monograph. However, above stated parabolic PDEs with "maxima" are not profoundly studied. The stability as well blow-up phenomena of the solutions to functional parabolic PDEs with "maxima" were investigated in [3, 6, 7].

The main methods for assessment of solutions of functional PDEs as well for investigation of existence and uniqueness, stability, blow-up, asymptotic behavior, etc. are the Monotone-Iterative techniques ([2, 5]) and the Method of Generalized Quasilinearization, [8].

In Paragraph 2 we recall the basic definitions, hypotheses and preliminary notes connected with the solvability, stability and steady states to the functional PDEs. In Paragraph 3 are considered monotonicity and existence-comparison results. The stability results are considered in Paragraph 4. The uniqueness of the solution is discussed in Paragraph 5.

2 Preliminary notes.

Denote the partial derivative $u_t \equiv \frac{\partial u}{\partial t}$, that stands for the evolution rate of the unknown density $u = u(t, x)$, that means concentration, temperature, population, etc. In most cases the evolutionary model is described by an IBVP like (1), where the unknown function u starts in some fixed initial moment t_0 and after passing a finite period of time describes the changes in the density. The basic question arising from the equations with "maxima" is whether, as time t increases, the solution $u = u(t, x)$ remains in a neighborhood of a steady-state solution $u_s = u_s(x)$. The second question is whether the solution $u(t, x)$ converges to the steady state (steady-state solution) as $t \to +\infty$. It is important to know for a given steady state u_s what is the set of initial functions whose corresponding time-dependent solutions converge to u_s as $t \to +\infty$. This leads to the problem of stability of a steady-state solution, often called Lyapunov stability. The asymptotic stability of such a solution and its stability region also would be of interest.

Suppose that $0 < \theta < 1$. Then the map $u: \Theta \subset \mathbb{R}^{n+1} \to \mathbb{R}$ is said to be θ Hölder (with a constant κ) if $|u(x) - u(y)| \leq \kappa \parallel x - y \parallel^{\theta}$. We write $u \in C^{1+\theta}$ if $u(\cdot)$ admits partial derivatives which are θ Hölder.

Introduce the following notations:

$D_T \equiv (0, T] \times \Omega, \qquad S_T \equiv (0, T] \times \partial\Omega, \qquad D_{-\sigma} \equiv [-\sigma, 0] \times \Omega,$
$\mathcal{CD} \equiv \mathbb{R}^+ \times \Omega, \qquad E_T \equiv [-\sigma, T] \times \overline{\Omega}.$

There exists in the technical applications a functional PDE having the form

$$\begin{aligned} &(a)\ \ u_t - Lu = f(t, x, u(t, x)) + R(t, x, u(t - \sigma, x)) \ \ in \ \ D_T, \\ &(b)\ \ Bu = h(t, x) \ \ on \ \ S_T, \\ &(c)\ \ u(t, x) = \eta_0(t, x) \ \ in \ \ D_{-\sigma}, \end{aligned} \qquad (2)$$

where $T > 0$, $\sigma > 0$, the function $\eta_0(t, x)$ is known nonnegative and Hölder continuous in $D_{-\sigma}$ with initial function $u_0(x) = \eta_0(0, x) \in C^{\theta}(\overline{\Omega})$, $h(t, x)$ is assumed in the class $C^{1+\theta}(S_T)$. Further the operator

$$L \equiv \sum_{i,j=1}^{n} a_{ij}(t, x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} b_j(t, x) \frac{\partial}{\partial x_j} \qquad (3)$$

in (2)(a) is uniformly elliptic in the sense that the matrix $\{a_{ij}(t, x)\}$ is positive definite on $\overline{\Omega}$ with constant not depending on t. We assume that the coefficients of L are in the class $C^{1+\theta}(\overline{\Omega})$ $(0 < \theta < 1)$. The boundary operator B is defined by $B \equiv \alpha_0(t, x) \frac{\partial}{\partial \nu} + \beta_0(t, x)$, where $\alpha_0(t, x)$ and $\beta_0(t, x)$ are nonnegative functions in $C^{1+\theta}(\partial\Omega)$ for $t \in [0, \infty)$ and not identically zero on $[0, \infty) \times \partial\Omega$; $\partial / \partial \nu$ is the outward normal derivative on $\partial\Omega$. Both functions f and R are Hölder continuous in $D_T \times \mathbb{R}$. In addition, $f(t, x, \eta)$ and $R(t, x, \eta)$ are assumed to be C^1-functions in η.

Consider the IBVP with "maxima"

$$
\begin{aligned}
&(a)\ u_t - Lu = f(t,x,u(t,x)) + R(t,x, \max_{s\in[t-\sigma,t]} u(s,x)) \quad in \quad D_T,\\
&(b)\ Bu = h(t,x) \quad on \quad S_T,\\
&(c)\ u(t,x) = \eta_0(t,x) \quad in \quad D_{-\sigma},
\end{aligned} \tag{4}
$$

where σ is a given positive constant representing the delay by which is determined the third argument $\max_{s\in[t-\sigma,t]} u(s,x)$ of the function R, and $\eta_0(0,x) = u_0(x)$ in Ω. Assume that f, $R \in C^{1+\theta}$. A solution $u(\cdot,\cdot)$ of IBVP belongs to the class $C^{1,2}(D_T)$, i.e. $u(\cdot,x) \in C^1$ and $u(t,\cdot) \in C^2$, when it satisfies (4).

Recall some basic definitions.

Definition 1 *Let the function $R(\cdot,\cdot,\eta)$ be monotone nondecreasing in η. A function $u \in C^{\theta}(E_T) \cap C^{1,2}(D_T)$ is called an upper solution of IBVP (4) if:*

$$
\begin{aligned}
&(a)\ \tilde{u}_t - L\tilde{u} \geq f(t,x,u(t,x)) + R(t,x, \max_{s\in[t-\sigma,t]} \tilde{u}(s)) \quad in \quad D_T,\\
&(b)\ B\tilde{u} \geq h(t,x) \quad on \quad S_T,\\
&(c)\ \tilde{u}(0,x) \geq \eta_0(t,x) \quad in \quad D_{-\sigma}.
\end{aligned} \tag{5}
$$

Similarly, $\hat{u} \in C^{\theta}(E_T) \cap C^{1,2}(D_T)$ is called a lower solution if it satisfies the reversed inequalities in (5).

Definition 2 *A pair $\tilde{u} = \tilde{u}(t,x)$, $\hat{u} = \hat{u}(t,x)$ is called ordered if $\tilde{u} \geq \hat{u}$ in E_T. Then the set of all functions $z = z(t,x)$ such that $\hat{u} \leq z \leq \tilde{u}$ in E_T is denoted by $\langle \hat{u}, \tilde{u} \rangle$ and is called sector.*

There exist mixed parabolic problems with solutions which do not depend on the time t. Such solutions call steady state solutions or **steady-states**. We denote these solutions by $u_s = u_s(x)$.

Definition 3 *A steady state solution $u_s(x)$ of (4) is said to be stable if for arbitrary taken positive number ε there exists another positive number δ such that*

$$|u(t,x) - u_s(x)| < \varepsilon \quad in \quad D_T \tag{6}$$

whenever $|u_0(x) - u_s(x)| < \delta$ in Ω, where $u_0(x) = \eta_0(0,x)$. If the problem (4) is defined in CD instead D_T, and in addition

$\lim_{t\to\infty} |u(t,x) - u_s(x)| = 0, \ \ uniformly \ \ on \ \ \overline{\Omega}$, (7)

then u_s is said to be asymptotically stable. The steady-state solution u_s is called unstable if it is not stable.

In terms of the sup-norm in the space of continuous functions $C(\overline{\Omega})$ the condition (6) is equivalent to

$$\| u - u_s \|_0 = \sup_{x\in\overline{\Omega}} |u(t,x) - u_s(x) < \varepsilon \quad for \ \ every \quad t > 0 \tag{8}$$

whenever $\| u_0 - u_s \|_0 < \delta$, and condition (7) for asymptotic stability becomes

$$\lim_{t\to\infty} \| u - u_s \|_0 = 0. \tag{9}$$

The above definition implies that if u_s is asymptotically stable then it is an isolated steady-state solution in the sense that there is a neighborhood $\mathcal{U}$ of u_s in $C(\overline{\Omega})$ such that u_s is the only steady-state solution in $\mathcal{U}$.

Definition 4 *A steady state solution u_s is said to be exponentially asymptotically stable when conditions (6) and (7) hold and the convergence in (7) is in exponential order. In other words, there exist positive constants ρ, α such that*

$|u(t,x) - u_s(x)| \leq \rho e^{-\alpha t}, \;\; for \;\; t > 0, \; x \in \overline{\Omega}$,(10)

whenever it holds at $t = 0$.

Definition 5 *The set of initial functions $\eta_0(t,x)$ defined in $[-\sigma, 0] \times \Omega$ under condition $u_0(x) = \eta_0(0,x)$ for $x \in \Omega$ whose corresponding solutions $u(t,x)$ of (4) satisfy conditions (6) and (7) is called stability region of u_s. If it is true for all the initial functions then u_s is said to be globally asymptotically stable.*

Assume that following hypotheses are satisfied:

(H1) $f(t,x,0) = R(t,x,0) = 0$ for $(t,x) \in D_T$ and $h(t,x) = 0$, $\beta_0(x) \neq 0$ for $(t,x) \in S_T$.

Let $\lambda(t)$ and $\Phi(x)(t)$ in $\overline{\Omega}$ are the principal eigenvalue and correspondent normalized eigenfunction, respectively, of the elliptical problem

$$\begin{aligned} -Lu &= \lambda u \;\; in \;\; \Omega, \\ Bu &= 0 \;\; on \;\; \partial\Omega. \end{aligned} \tag{11}$$

We note that $\Phi(x)(\cdot)$ is always normalized by $\max\{\Phi(x)(t) \colon \; x \in \overline{\Omega}\} = 1$, and $\lambda(t) \geq \lambda_0 > 0$.

(H2) There is a positive number $\alpha < \lambda_0$ such that

$$f(t,x,\eta) \leq (\lambda_0 - \alpha)\eta \;\; for \;\; \eta \geq 0 \;\; and \;\; (t,x) \in D_T. \tag{12}$$

(H3) There is a continuous function $\gamma(\sigma,t)$ defined by $\gamma \colon \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \Gamma$ and such that

$$R(t,x,\xi) \leq \gamma(\sigma,t)\xi \;\; for \;\; \xi > 0 \;\; and \;\; (t,x) \in D_T, \tag{13}$$

where Γ is some bounded real subset of $\mathbb{R}^+$. First of all, we prove the following elementary result:

(H4) The partial derivative $f_\eta(t,x,\eta)$ of the function $f(t,x,\eta)$ satisfies the estimate

$$f_\eta(t,x,\eta) \le \lambda_0 - \alpha$$
$$for \quad |\eta| \le \rho,\ \rho > 0\ \ ((t,x) \in D_T), \tag{14}$$

where $\lambda_0 > \alpha > 0$.

(H5) The partial derivative $R_\xi(t,x,\xi)$ of the function $R(t,x,\xi)$ satisfies the estimate

$$R_\xi(t,x,\xi) \le \lambda_0 - \beta$$
$$for \quad |\xi| \le \rho_1,\ \rho_1 > 0\ \ ((t,x) \in D_T), \tag{15}$$

where $\lambda_0 > \beta > 0$.

(H6) Let in the sector $\langle \hat{u}, \tilde{u} \rangle$ we assume that there exist bounded functions $\underline{c} \equiv \underline{c}(t,x)$ and $\overline{c} \equiv \overline{c}(t,x)$ such that for the reaction function $F = f + R$ in (4) the following inequalities hold true,

$$-\underline{c}(u_1 - u_2) \le F(t,x,u_1) - F(t,x,u_2) \le \overline{c}(u_1 - u_2), \tag{16}$$

where $\hat{u} \le u_2 \le u_1 \le \tilde{u}$ $((t,x) \in D_T)$.

Remark 1 *The multipliers $\underline{c}$, $\overline{c}$ stated in (16) can be defined as it is in [3],* $\underline{c}(t,x) \equiv \sup\{-f_u(t,x,u) : \hat{u} \le u \le \tilde{u}\}$, $\overline{c}(t,x) \equiv \sup\{f_u(t,x,u) : \hat{u} \le u \le \tilde{u}\}$.

Define the function

$$F_1(t,x,u) \equiv \underline{c}(t,x)u + f(t,x,u). \tag{17}$$

Obviously, the function F_1 is Hölder continuous in $\overline{D}_T \times \langle \hat{u}, \tilde{u} \rangle$ and is monotone nondecreasing in $u \in \langle \hat{u}, \tilde{u} \rangle$. Also F_1 satisfies the Lipschitz condition

$$|F_1(t,x,u_1) - F_1(t,x,u_2)| \le K|u_1 - u_2| \quad for \quad u_1, u_2 \in \langle \hat{u}, \tilde{u} \rangle, \tag{18}$$

where for instance K may be taken as an upper bound of $|\underline{c}(t,x)| + |\overline{c}(t,x)|$ in $\overline{D}_T$.

Further we use the following lemma,

Lemma 1 *Under (H4) and (H5) the functions f and R satisfy:*

$$f(t,x,\eta) \le (\lambda_0 - \alpha)\eta$$
$$for \quad |\eta| \le \rho,\ \rho > 0\ \ ((t,x) \in D_T) \tag{19}$$

and

$$fR(t,x,\xi) \le (\lambda_0 - \beta)\xi$$
$$for \quad |\xi| \le \rho_1,\ \rho_1 > 0\ \ \big((t,x) \in D_T\big), \tag{20}$$

respectively.

3 Monotonicity and existence-comparison results.

Define the following linear differential operator of parabolic type,

$$\mathbb{L}_c \equiv \left(\frac{\partial}{\partial t} - L + c\right) \quad in \quad (0,T]\times\mathbb{R}^n, \tag{21}$$

where L and T are the same as those in (4), and $c = c(t,x)$ is a bounded function in $(0,T]\times\mathbb{R}^n$. Consider a pair of ordered upper and lower solutions to the problem (4) $\tilde{u}$ and $\hat{u}$, respectively, and use $u^{(0)} = \tilde{u}$ and $u^{(0)} = \hat{u}$ as two independent initial iterations and define the iteration process

$$\begin{aligned} &(a)\;\; \mathbb{L}_c[u^{(k)}] = \underline{c}u^{(k-1)} + f(t,x,u^{(k-1)}(t,x)) + R(t,x,\max_{s\in[t-\sigma,t]} u^{(k-1)}(s,x)) \quad in \quad D_T,\\ &(b)\;\; Bu^{(k)} = h(t,x) \quad on \quad S_T,\\ &(c)\;\; u^{(k)}(t,x) = \eta_0(t,x) \quad in \quad D_{-\sigma}, \end{aligned} \tag{22}$$

where $\underline{c} = \underline{c}(t,x)$ is some continuous function that can be taken as $\underline{c} = \sup_u\{-f_u(t,x,u): \hat{u} \le u \le \tilde{u}\}$. Refer to the sequences $\{\bar{u}^{(k)}\}$, $\{\underline{u}^{(k)}\}$ as upper and lower sequences, respectively.

Lemma 2 (Lemma for monotonicity, [9]) *Let $R(t,x,\xi)$ be monotone nondecreasing in $\xi \in \langle\hat{u},\tilde{u}\rangle$. Then the sequences $\{\overline{u}^{(k)}\}$, $\{\underline{u}^{(k)}\}$ given by (22) with $\overline{u}^{(0)} = \tilde{u}$ and $\{\underline{u}^{(0)}\} = \hat{u}$ are well defined and possess the monotone property $\hat{u} \le \underline{u}^{(k)} \le \underline{u}^{(k+1)} \le \overline{u}^{(k+1)} \le \overline{u}^{(k)} \le \tilde{u} \quad in \quad E_T$*

Let define the functions

$$q_1^{(k)}(t,x) = R(t,x,\max_{s\in[t-\sigma,t]} u^{(k)}(s,x)), \quad in \quad (t,x)\in D_T, \tag{23}$$

and

$$q_2^{(k)}(t,x) = \underline{c}(t,x)u^{(k)}(t,x) + f(t,x,u^{(k)}(t,x)), \tag{24}$$

where $\{u^{(k)}\}$ is the sequence from (22) with initial function $\{u^{(0)}\} \in C^\theta(E_T)$ and $u^{(k)} \in C^\theta(D_T)$. By the Hölder continuity of $R(t,x,\xi)$ and the Lipschitz condition of R in ξ we conclude that both functions $R(t,x,u^{(0)})$ and $q_2^{(k)}(t,x)$ are Hölder continuous in D_T with the same exponent θ, whenever $u^{(k)} \in C^\theta(D_T)$. In (22) we use the initial function $u^{(1)}(0,x) = \eta_0(0,x) = u_0(x)$, such that the solution $u^{(1)}(t,x)$ exists in $C^\theta(D_T)$.

The following theorem from [9] gives us an existence-comparison result that is very important for further study of the problem under consideration.

Theorem 1 *Let $\tilde{u}$, $\hat{u}$ be ordered upper and lower solutions of (4), and let $f(t,x,\xi)$, $R(t,x,\xi)$ be C^1-functions of ξ and $\partial R/\partial\xi \ge 0$ for $\xi \in \langle\hat{u},\tilde{u}\rangle$. Then the sequences $\{\overline{u}^{(k)}\}$, $\{\underline{u}^{(k)}\}$ given by (22) converge monotonically to a unique solution $u = u(t,x)$ of (4), and $\hat{u} \le u \le \tilde{u}$ in E_T.*

Next we quote an existence and uniqueness result that can be seen also in [9].

Theorem 2 *Let $\tilde{u}$, $\hat{u}$ be ordered upper and lower solutions of (4), and let $F = f + R$ satisfies (H6). Then there exist sequences $\{\overline{u}^{(k)}\}$, $\{\underline{u}^{(k)}\}$ which converge monotonically to a unique solution u of (4) and*

$\hat{u} \le \underline{u}^{(k)} \le \underline{u}^{(k+1)} \le u \le \overline{u}^{(k+1)} \le \overline{u}^{(k)} \le \tilde{u} \quad in \quad E_T.$

Here we write E_T instead D_T.

4 Stability result.

Here we establish some stability criteria for (4). The proof in detail one can find in [3, 6, 7].

Theorem 3 *Let $f(t,x,\xi)$ and $R(t,x,\xi)$ be C^1-functions w.r.t. $\xi \in R^+$ and let the conditions (H1)-(H3), (H6) and the inequality*

$$0 < \gamma(\sigma,t) \le Ae^{(-\alpha+A)\sigma}, \;\; for \;\; t \ge 0, \;\; \sigma > 0 \text{ and } \alpha > \mathrm{A} \;\; (A = const)$$

be satisfied. Then a unique nonnegative solution $u = (t,x)$ to (4) exists. Furthermore if $0 \le u(0,x) \le \rho\Phi(x)$ then

$$0 < u(t,x) \le \rho e^{(-\alpha+A)t}\Phi(x), \;\; (t,x) \in E_T,$$

whenever it holds at $t = 0$ (A is a constant).

Theorem 4 *Given the problem (4). Let the hypotheses in Theorem 3 hold except that the condition (H2), (H3) are replaced by (H4), (H5), respectively, and (H6) in addition. If σ satisfies*

$$0 < \sigma \le \frac{1}{\alpha - \mathrm{A}} \ln \frac{\mathrm{A}}{\lambda_0 - \beta},$$

where $\mathrm{A} = \mathrm{A}(\beta,\sigma) \in (\lambda_0 - \beta, \alpha)$ depends continuously on β and σ, then there exists a solution $u = u(t,x)$ of (4) that satisfies the estimate

$$|u(t,x)| \le \rho e^{(-\alpha+A)t}\Phi(x), \;\; t > 0, \; x \in \overline{\Omega}$$

whenever it holds at $t = 0$. If assume $T = +\infty$ and consider the problem (4) in CD, then the trivial solution $u_s \equiv 0$ is exponentially asymptotically stable.

Concluding our note we emphasise that the following inequality hold:

$$\alpha > \mathrm{A} \ge (\lambda_0 - \beta)e^{(\alpha - \mathrm{A})\sigma} \ge \lambda_0 - \beta,$$

hence

$2\lambda_0 > \alpha + \beta > \lambda_0$ and $1 < \dfrac{\alpha+\beta}{\lambda_0} < 2.$

(H7) Let the reaction function F be in the form $F(t,x,u(t,x), \max\limits_{s\in[t-\sigma,t]} u(s,x)) \equiv f(t,x,u(t,x)) + R(t,x, \max\limits_{s\in[t-\sigma,t]} u(s,x))$, and $f(t,x,\cdot)$ belongs to $L_{loc}(\mathbb{R}^+)$ (the set of all locally Lipschitzean functions on $\mathbb{R}^+$), while $R(t,x,\cdot)$ is bounded on the bounded sets, where $(t,x)\in D_T$

Theorem 5 *Suppose **(H7)** hold and $u(t,x)$ is the nonnegative solution of (4) provided that the reaction function F has the form*

$F\left(t,x,u(t,x), \max\limits_{s\in[t-\sigma,t]} u(s,x)\right) \equiv f\big(t,x,u(t,x)\big) + R\left(t,x, \max\limits_{s\in[t-\sigma,t]} u(s,x)\right)$. *If there exist constants $\alpha > 0$, $\beta > 0$ and $A > 0$ such that*

$$\begin{aligned}&(a)\ \ f(t,x,\eta) \ge \left(\frac{\lambda_0}{2} + \alpha + \frac{A}{2}\right)\eta \ \ \forall\ \ \eta \ge 0,\\ &(b)\ \ R(t,x,\xi(x)) \ge \left(\frac{\lambda_0}{2} + \beta + \frac{A}{2}\right)\|\xi(x)\| \ \ \forall\ \ \xi(x)\in C([-\sigma,0],\mathbb{R}),\end{aligned} \tag{25}$$

then $\forall \delta > 0$ and $\forall \eta_0(t,x) \ge \delta$ in $D_{-\sigma}$ the solution of (4)

$$u(t,x) \ge \delta e^{(\alpha+\beta+A)t}\Phi(x). \tag{26}$$

Theorem 6 *Suppose **(H1)** hold and $u(t,x)$ is the nonnegative solution of (4) provided that the reaction function F has the form* $F\left(t,x,u(t,x), \max\limits_{s\in[t-\sigma,t]} u(s,x)\right) \equiv f\big(t,x,u(t,x)\big) + R\left(t,x, \max\limits_{s\in[t-\sigma,t]} u(s,x)\right)$. *If there exist constants $\alpha > 0$, $\beta > 0$ and $A > 0$ such that*

$$\begin{aligned}&(a)\ \ f(t,x,\eta) \ge \left[\left(\alpha + \frac{A}{2}\right)(1+t)^{-1} + \frac{\lambda_0}{2}\right]\eta \ \ \forall\ \eta \ge 0,\\ &(b)\ \ R(t,x,\xi(x)) \ge \left[\left(\beta + \frac{A}{2}\right)(1+t)^{-1} + \frac{\lambda_0}{2}\right]\|\xi(x)\|\\ &\quad \text{and}\ \ \|\xi(x)\| = \max_{s\in[-\sigma,T-\sigma]} \xi(x)(s),\end{aligned} \tag{27}$$

then $\forall \delta > 0$ and for any $\eta_0(t,x) \ge \delta$

$$u(t,x) \ge \delta(1+t)^{\alpha+\beta+A}\Phi(x)\ \ \textit{for}\ \ t > 0, x\in\overline{\Omega}. \tag{28}$$

Theorem 7 *Let* **(H1)** *hold and let* z *be a nonnegative function defined on* $[0,T_0)\times\overline{\Omega}$ *and unbounded at some point in* $\overline{\Omega}$ *as* $t\to T_0$*. If* z *is a lower solution of (4) in* $\overline{D}_T$ *for every* $T<T_0$ *then there exists other positive number* $T^*\le T_0$ *such that a unique solution* $u=u(t,x)$ *exists in* $(0,T_0]\times\overline{\Omega}$ *and* $\lim\limits_{t\to T^*}[\max\limits_{x\in\overline{\Omega}} u]=\infty$*.*

Lemma 3 *Let the condition* $m(t)=(1+t)^{-1}\beta(t),\ 0<\beta(t)\le A(1-\sigma)^{\alpha-A}$ *for all* $t\in[0,T).$ *be satisfied. Then the function* $z=\rho(1+t)^{-\alpha+A}$ *satisfies the differential inequality*

$$\frac{dz}{dt}\ge-\alpha(1+t)^{-1}z+m(t)\max_{s\in[t-\sigma,t]}z(s),\ t\in[0,T). \qquad (28^*)$$

Proof. We have $\frac{dz}{dt}=(A-\alpha)\rho(1+t)^{-\alpha+A-1}$. Thus the differential inequality (28*) becomes:

$$(A-\alpha)\rho(1+t)^{-\alpha+A-1}\ge-\rho\alpha(1+t)^{-1}(1+t)^{-\alpha+A}+\rho m(t)(1+t-\sigma)^{-\alpha+A},\,t\in[0,T).$$

Suppose (28*) is not true, hence

$$A(1+t)^{-\alpha+A-1}<m(t)(1+t-\sigma)^{-\alpha+A},\quad and$$

$$m(t)>A(1+t)^{-1}\left(1-\frac{\sigma}{1+t}\right)^{\alpha-A}.$$

Having in mind this and also $m(t)=(1+t)^{-1}\beta(t),$ (29)

$$0<\beta(t)\le A(1-\sigma)^{\alpha-A}\ \ for\ all\ t\in[0,T), \qquad (30)$$

$$0<\beta(t)\le A\left(1-\frac{\sigma}{1+t}\right)^{\alpha-A}\ \ for\ t\in[0,T). \qquad (31)$$

(29) - (31) it turns out that the latter contradicts to (31).

5 Uniqueness of the solutions.

First, we consider the regularity of $U(t,x) \equiv \max\limits_{s\in[t-\sigma,t]} u(s,x)$.

Lemma 4 *If* $u \in Lip^{\theta}(E_T)$ *then* $(t,x) \to \max_{s\in[t-\sigma,t]} u(s,x)$ *is in* $Lip^{\theta}(\overline{D_T})$.

Proof. Let us choose any points $(t_1,x_1),(t_2,x_2) \in [0,T]\times\overline{\Omega}$. Since

$$|u(t_1+s,x_1)-u(t_2+s,x_2)|+u(t_2+s,x_2) \ge u(t_1+s,x_1),$$

we have

$$\max_{s\in[-\sigma,0]} |u(t_1+s,x_1)-u(t_2+s,x_2)| + \max_{s\in[-\sigma,0]} u(t_2+s,x_2) \ge \max_{s\in[-\sigma,0]} u(t_1+s,x_1).$$

Similarly we have

$$\max_{s\in[-\sigma,0]} |u(t_1+s,x_1)-u(t_2+s,x_2)| + \max_{s\in[-\sigma,0]} u(t_1+s,x_1) \ge \max_{s\in[-\sigma,0]} u(t_2+s,x_2).$$

Then we get

$$\max_{s\in[-\sigma,0]} |u(t_1+s,x_1)-u(t_2+s,x_2)| \ge \left| \max_{s\in[-\sigma,0]} u(t_1+s,x_1) - \max_{s\in[-\sigma,0]} u(t_2+s,x_2) \right|. \tag{32}$$

We have by admission that

$$|u(t_1,x_1)-u(t_2,x_2)| \le H(|t_1-t_2|+|x_1-x_2|)^{\theta},$$

where H is the Hölderian constant which is independent of t_1,t_2,x_1 and x_2.

Due to (32) one has that

$$\frac{|U(t_1,x_1)-U(t_2,x_2)|}{(|t_1-t_2|+|x_1-x_2|)^{\theta}} \le \max_{s\in[-\sigma,0]} \frac{|u(t_1+s,x_1)-u(t_2+s,x_2)|}{(|t_1-t_2|+|x_1-x_2|)^{\theta}} \le H,$$
$$for \quad t_1,t_2 \in [0,T], \; x_1,x_2 \in \overline{\Omega}.$$

Hence $U(t,x)$ is in $Lip^{\theta}(D_T)$.

Remark 2 *Evidently Lemma 4 is true when Hölder is replaced with Lipschitz. However* $(t,x) \to \max_{s\in[t-\sigma,t]} u(s,x)$ *is not continuously differentiable (but only Lipschitz) even for analitic* $u(\cdot,\cdot)$. *Indeed let* $u(x)=x^2$. *If* $t<\dfrac{\sigma}{2}$, *then* $\max_{s\in[x-\sigma,x]} u(s)=(x-\sigma)^2$. *When* $x \ge \dfrac{\sigma}{2}$, *then* $\max_{s\in[x-\sigma,x]} u(s)=x^2$. *The left derivative of* $\max_{s\in[x-\sigma,x]} u(s)$ *at* $\dfrac{\sigma}{2}$ *is* $-\sigma$ *while the right one is* σ.

We notice that f and R are C^1-functions in the sector $\langle \hat{u},\tilde{u} \rangle$. For a given pair of ordered upper and lower solutions $\tilde{u}$, $\hat{u}$, we use $u^{(0)}=\tilde{u}$ and $u^{(0)}=\hat{u}$ as two independent initial iterations and construct their respective sequences from the iteration process

$$u_t^{(k)} - Lu^{(k)} + cu^{(k)} = cu^{(k-1)} + f(t,x,u^{(k-1)}) + R(t,x,\max_{s\in[t-\sigma,t]} u^{(k-1)}(s,x)) \text{ in } D_T$$

$$Bu^{(k)} = h(t,x) \quad \text{on } S_T$$

$$u^{(k)}(t,x) = \eta_0(t,x) \quad \text{in } D_{-\sigma},$$

where $c(t,x) = \sup\{-f_u(t,x,u); \hat{u} \le u \le \widetilde{u}\}$. Denote the above stated sequences by $\{\overline{u}^{(k)}\}$, and refer to them as upper and lower sequences, respectively. The following statement hold:

Theorem 8 ([3], [9]) *Under the above assumptions, the sequences* $\{\overline{u}^{(k)}\}$, $\{u^{(k)}\}$ *converge monotonically to a unique solution* u *to (4), and* $\hat{u} \le u \le \widetilde{u}$ *in* E_T.

The uniqueness can be formulated by the following statement:

Theorem 9 *Let the hypotheses (H1)-(H3) be satisfied. Then a unique solution* $u = u(t,x)$ *of (4) exists and satisfies the inequality*

$$|u(t,x)| \le \rho(1+t)^{-\alpha+A}\Phi(x),\ (t,x) \in E_T, \tag{30}$$

whenever $|\eta_0(t,x)| \le \rho(1+t)^{-\alpha+A}\Phi(x)$ in $D_{-\sigma}$. And the steady-state solution $u \equiv 0$ is asymptotically stable.

Conclusion.

Notice that if we study the problem (4) in CD instead of D_T, then the solution $u(.,x)$ decays uniformly on x. Another result of interest for existence and uniqueness can be obtained under analogical requirements, that is, a unique solution $u = u(t,x)$ of (4) exists and satisfies the inequality

$$|u(t,x)| \le \rho(1+t)^{-\alpha+\mathrm{A}}\Phi(x) \quad (t,x) \in E_T$$

whenever $|\eta_0(t,x)| \le \rho(1+t)^{-\alpha+\mathrm{A}}\Phi(x)$ in $D_{-\sigma}$, and the steady-state solution $u \equiv 0$ is asymptotically stable. We refer the reader to [3, 6, 7] for details.

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